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## Week ~~8~~ Thursday - Optimization & Limit computing

*Steps for optimization problems: (2 keys!)*

① Draw & Label

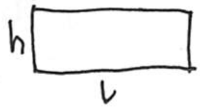
★② Constraint

★③ Formula for the something we want to optimize

④ domain

⑤ Find global max/min & Justify by table of $f'$.

1. *Above steps must be quite abstract. Let's look at one easier example. (problem from textbook)*
   Which rectangle of area 100 in$^2$ minimizes its $\underset{h}{\text{height}}$ plus two times its $\underset{l}{\text{length}}$?



Constraint:

$hl = 100$

$\Rightarrow h = \frac{100}{l}$

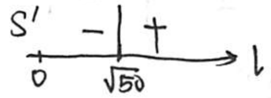
Formula

$S = h + 2l$

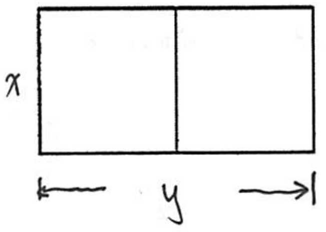
$= \frac{100}{l} + 2l \quad (l > 0)$

$S' = -\frac{100}{l^2} + 2 = 0 \Rightarrow$ critical pts: $l = \sqrt{50}$

$S = \frac{100}{\sqrt{50}} + 2\sqrt{50}$.



2. *It's time for you to try! (from midterm2 of Fall 2014 Lec003)*
   A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. Find the dimensions of the fenced area that will minimize the amount of the fencing material used.



Constraint:

$xy = 1,500,000$

$y = \frac{1,500,000}{x}$

Formula

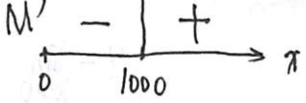
$M = 3x + 2y$

$= 3x + \frac{3,000,000}{x} \quad (x > 0)$

$M' = 3 - \frac{3,000,000}{x^2}$

$\Rightarrow$ critical pts: $x = 1000$

$y = \frac{1,500,000}{1000} = 1500$



When height is 1000 ft, width is 1500 ft, the material is minimized.

**Properties of Exponentials and logorithms:**

(1) $e^a e^b = e^{a+b}$ $\quad \frac{e^a}{e^b} = e^{a-b}$

(2) $\ln a + \ln b = \ln(ab)$ $\quad \ln a - \ln b = \ln \frac{a}{b}$

(3) $e^{\ln a} = a$ $\quad \ln(e^a) = a$

3. Limit computing involving exponentials and logorithms

   *(the following problems are modified from our textbook; similar problems appear in hw8 as well)*

   (a) $\lim_{x\to 1} \frac{\arctan(x-1)}{\ln x}$

   (b) $\lim_{x\to\infty} x^3 e^{-\frac{1}{x}}$

   (c) $\lim_{t\to\infty} \frac{e^t+e^{-t}}{2e^t-te^{-t}}$

   (d) $\lim_{x\to\infty} \frac{e^{-x}-e^{-\frac{x}{2}}}{\sqrt{e^x+1}}$

   (e) $\lim_{x\to\infty} \frac{\ln(x^2)}{\ln x}$

   (f) $\lim_{x\to\infty} \ln(x+2) - \ln x$

   (g) $\lim_{x\to\infty} (x+\pi)^{\frac{1}{x+1}}$

(a) "Plug in": "$\frac{0}{0}$" $\Rightarrow$ L'Hopital

$$? = \lim_{x\to 1} \frac{\frac{1}{1+(x-1)^2}}{\frac{1}{x}} = \frac{1}{1} = 1$$

(b) Plug in "$\infty \cdot 1$"

$? = $ DNE

(c) Plug in "$\frac{\infty}{\infty}$" highest order : $e^t$

$$? = \lim_{t\to\infty} \frac{e^t+e^{-t}}{2e^t-te^{-t}} \frac{\frac{1}{e^t}}{\frac{1}{e^t}} = \lim_{t\to\infty} \frac{1+\frac{1}{e^{2t}}}{2-\frac{t}{e^{2t}}} = \frac{1}{2}$$

(d) Plug in "$\frac{0}{\infty}$"

$? = 0$

(e) Plug in "$\frac{\infty}{\infty}$" can simplify!

$$? = \lim_{x\to\infty} \frac{2\ln x}{\ln x} = 2$$

(f) $\lim_{x\to\infty} \ln \frac{x+2}{x} = \ln 1 = 0$

g) $L = \lim_{x\to\infty} (x+\pi)^{\frac{1}{x+1}}$

$$\ln L = \lim_{x\to\infty} \frac{\ln(x+\pi)}{x+1} = \lim_{x\to\infty} \frac{\frac{1}{x+\pi}}{1} = 0 \Rightarrow L = e^0 = 1$$

*(Optional: 2 crazy-looking problems from past exams - quite hard, but could be a good practice.)*

(h) $\lim_{x\to\infty} \frac{e^x \ln x + x^{10}}{xe^x - x^{20}e^{-x}}$

(i) $\lim_{x\to 0} \frac{1-\cos x}{\ln(x+1)\cdot(e^x+1)}$

(h) "$\frac{\infty}{\infty}$" highest order : $xe^x$

$$? = \lim_{x\to\infty} \frac{e^x \ln x + x^{10}}{xe^x - x^{20}e^{-x}} \frac{1/(xe^x)}{1/(xe^x)}$$

$$= \lim_{x\to\infty} \frac{\frac{\ln x}{x} + \frac{x^9}{e^x}}{1 - \frac{x^{19}}{e^{2x}}} = \frac{0+0}{1-0} = 0$$

(i) "$\frac{0}{0}$" L'Hopital

$$? = \lim_{x\to 0} \frac{\sin x}{\frac{1}{x+1}(e^x+1) + \ln(x+1)\cdot e^x}$$

$$= \frac{0}{1\cdot 2 + 0} = 0$$